

Polar codes for q -ary channels, $q = 2^r$

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Abstract—We study polarization for nonbinary channels with input alphabet of size $q = 2^r, r = 2, 3, \dots$. Using Arıkan’s polarizing kernel H_2 , we prove that the virtual channels that arise in the process of polarization converge to q -ary channels with capacity $1, 2, \dots, r$ bits, and that the total transmission rate approaches the symmetric capacity of the channel. This leads to an explicit transmission scheme for q -ary channels. The error probability of decoding using successive cancellation behaves as $\exp(-N^\alpha)$, where N is the code length and α is any constant less than 0.5.

I. INTRODUCTION

Polarization is a new concept in information theory discovered in the context of capacity-achieving families of codes for symmetric memoryless channels and later generalized to source coding, multi-user channels and other problems. Polarization was first described by Arıkan [1] who constructed binary codes that achieve capacity of symmetric memoryless channels (and “symmetric capacity” of general binary-input channels). The main idea of [1] is to combine the bits of the source sequence using repeated application of the “polarization kernel” $H_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. The resulting linear code of length $N = 2^n$ has the generator matrix which forms a submatrix of $G_N = BH_2^{\otimes n}$, where B is a permutation matrix. The choice of the rows of G_N is governed by the polarization of virtual channels for individual bits that arise in the process of channel combining and splitting. Namely, the data bits are written in the coordinates that correspond to near-perfect channels while the other bits are fixed to some values known to both the transmitter and the decoder. It was shown later that polarization on binary channels can be achieved using a variety of other kernels: in particular, any $m \times m$ matrix whose columns cannot be arranged to form an upper triangular matrix, achieves the desired polarization [2].

A study of polar codes for channels with nonbinary input was undertaken by Şaşıoğlu et al. [3], [4] and Mori and Tanaka [5]. For prime q , it suffices to take the kernel H_2 , while for nonprime alphabets, the kernel is time-varying and not explicit. Namely, for prime q , [3] showed that there exist permutations of the input alphabet such that the virtual channels for individual q -ary symbols become either fully noisy or perfect, and the proportion of perfect channels approaches the symmetric capacity, in analogy with the results for binary codes in [1]. At the same time, [3] remarks that the transmission scheme that uses the kernel H_2 with modulo- q addition for composite q does not necessarily lead to the polarization of the channels

to the two extremes. Rather, they show that there exists a sequence of permutations of the input alphabet such that when they are combined with H_2 , the virtual channels for the transmitted symbols become either nearly perfect or nearly useless.

The authors of [3] suggest several alternatives to the kernel H_2 that rely on randomized permutations or, in the case of $q = 2^r$, on multilevel schemes that implement polar coding for each of the bits of the symbol independently, combining them in the decoding procedure; see esp. [4].

In this paper we study polarization for channels with input alphabet of size $q = 2^r, r = 2, 3, \dots$. Suppose that the channel is given by a stochastic matrix $W(y|x)$ where $x \in \mathcal{X}, y \in \mathcal{Y}, \mathcal{X} = \{0, 1, \dots, q-1\}$, and \mathcal{Y} is a finite alphabet. Assuming that the channel combining is performed using the kernel H_2 with addition modulo q , we establish results about the polarization of channels for individual symbols. It turns out that virtual channels for the transmitted symbols converge to one of $r+1$ extremal configurations in which j out of r bits are transmitted near-perfectly while the remaining $r-j$ bits carry almost no information. Moreover, the good bits are always aligned to the right of the transmitted r -block, and no other situations arise in the limit. Thus, the extremal configurations for information rates that arise as a result of polarization are easily characterized: they form an upper-triangular matrix as described in Sect. II-B (see also Figs. 1, 2 in the final section of the paper). This characterization also constitutes the main difference of our results from the multilevel scheme in [4]: there, the set of extremal configurations can in principle have cardinality 2^r which complicates the code construction.

Another related work is the paper by Abbe and Telatar [6]. In it, the authors observed multilevel polarization in a somewhat different context. The main result of their paper provides a characterization of extremal points of the region of attainable rates when polar codes are used for each of the r users of a multiple-access channel. Namely, as shown in [6] (see also [7]), these points form a subset in the set of vertices of a matroid on the set of r users. [6] also remarks that these results translate directly to transmission over a q -ary DMC, showing that the rate polarizes to many levels. To explain the difference between [6] and our work we note that transmission over the multiple-access channel in [6] is set up in such a way that, once applied to the DMC, it corresponds to encoding each bit of the q -ary symbol by its own polar code (we again assume that $q = 2^r$). In other words, the polarization kernel

employed is a linear operator $G = I_r \otimes H_2$. Thus, the group acting on \mathcal{X} is $\mathbb{F}_{2^r}^+ = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ rather than the cyclic additive group of order q considered in this paper.

This work began as an attempt to construct polar codes for the *ordered symmetric channel*, introduced in our earlier paper [8]. This channel provides an information-theoretic model related to the ordered distance on binary r -vectors, defined as follows:

$$d_r(x, x') = \max\{j : x_j \neq x'_j\}, \quad \text{where } x, x' \in \{0, 1\}^r. \quad (1)$$

Below $\text{wt}_r(x) = d_r(x, 0)$ denotes the ordered weight of the symbol x . The ordered distance is an instance of a large class of metrics introduced in [9] following works of Niederreiter in numerical analysis [10]. It has subsequently appeared in a large number of works in algebraic combinatorics and coding theory; see e.g., [11] and references therein. We find it quite interesting that it independently arises in the study of polar codes on channels with input of size $q = 2^r$. Examples of q -ary polar codes for ordered symmetric channels can be easily constructed and analyzed.

Last but not least, when this work was in its final stages, we became aware of the paper by Sahebi and Pradhan [12] who also observed the multilevel polarization phenomenon for q -ary channels. At the same time, [12] did not give a proof of polarization, which constitutes the main technical part of our work. The motivation of the approach of [12] relates to a detailed study of linear and group codes on q -ary channels, and is also different from our approach.

In the next section we state and prove the main result, the convergence of the channels to one of the $r + 1$ extremal configurations, and deduce that polar codes achieve the symmetric capacity of the channel. Then we derive the rate of polarization and estimate the error probability of decoding, and give some examples.

II. POLARIZATION FOR q -ARY CHANNELS

We consider combining of the q -ary data under the action of the operator H_2 , where $q = 2^r, r \geq 2$. Let $W : \mathcal{X} \rightarrow \mathcal{Y}, |\mathcal{X}| = q$ be a discrete memoryless channel (DMC). The *symmetric capacity* of the channel W equals

$$I(W) \triangleq \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \frac{1}{q} W(y|x) \log \frac{W(y|x)}{\sum_{x' \in \mathcal{X}} \frac{1}{q} W(y|x')}$$

where the base of the logarithm is 2. Define the combined channel W_2 and the channels W^- and W^+ by

$$W_2(y_1, y_2 | u_1, u_2) = W(y_1 | u_1 + u_2) W(y_2 | u_2),$$

$$W^-(y_1, y_2 | u_1) = \sum_{u_2 \in \mathcal{X}} \frac{1}{q} W_2(y_1, y_2 | u_1, u_2), \quad (2)$$

$$W^+(y_1, y_2, u_1 | u_2) = \frac{1}{q} W_2(y_1, y_2 | u_1, u_2), \quad (3)$$

where u_1, u_2, y_1, y_2 are r -vectors and $+$ is a modulo- q sum. This transformation can be applied recursively to the channels W^-, W^+ resulting in four channels of the form $W^{b_1 b_2}, b_1, b_2 \in \{+, -\}$. After n steps we obtain $N = 2^n$

channels $W_N^{(j)}, j = 1, \dots, N$. For the case $q = 2$ it is shown in [1] that as n increases, the channels $W_N^{(j)}$ become either almost perfect or almost completely noisy (polarize). In formal terms, for any $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{|\{b \in \{+, -\}^n : I(W^b) \in (\varepsilon, 1 - \varepsilon)\}|}{2^n} = 0. \quad (4)$$

In this paper we extend this result to the case $q = 2^r, r > 1$.

As shown in [1], after n steps of the transformation (2)-(3) the channels $W_N^{(i)} : \mathcal{X} \rightarrow \mathcal{Y}^N \times \mathcal{X}^{i-1}, 1 \leq i \leq N$ are given by

$$W_N^{(i)}(y_1^N, u_1^{i-1} | u_i) = \frac{1}{q^{N-1}} \sum_{u_{i+1}^N \in \mathcal{X}^{N-i}} W^N(y_1^N | u_1^N G_N), \quad (5)$$

where $G_N = B H_2^{\otimes n}$ and B is a permutation matrix. Here we use the shorthand notation for sequences of symbols: for instance, $y_1^N \triangleq (y_1, y_2, \dots, y_N)$, etc.

A. Notation

For any pair of input symbols $x, x' \in \mathcal{X}$, the Bhattacharyya distance between them is

$$Z(W_{\{x, x'\}}) = \sum_{y \in \mathcal{Y}} \sqrt{W(y|x)W(y|x')}$$

where $W_{\{x, x'\}}$ is the channel obtained by restricting the input alphabet of W to the subset $\{x, x'\} \subset \mathcal{X}$.

Define the quantity $Z_v(W)$ for $v \in \mathcal{X} \setminus \{0\}$:

$$Z_v(W) = \frac{1}{2^r} \sum_{x \in \mathcal{X}} Z(W_{\{x, x+v\}}).$$

Introduce the i th average Bhattacharyya distance of the channel W by

$$Z_i(W) = \frac{1}{2^{i-1}} \sum_{v \in \mathcal{X}_i} Z_v(W) \quad (6)$$

where $i = 1, 2, \dots, r$ and $\mathcal{X}_i = \{v \in \mathcal{X} : \text{wt}_r(v) = i\}$. Then

$$Z(W) = \frac{1}{2^r(2^r - 1)} \sum_{x \neq x'} Z(W_{\{x, x'\}})$$

$$= \frac{1}{2^r - 1} \sum_{i=1}^r 2^{i-1} Z_i(W) \quad (7)$$

Recall the setting of [1] for the evolution of the channel parameters. On the set $\Omega = \{+, -\}^*$ of semi-infinite binary sequences define a σ -algebra \mathcal{F} on Ω generated by the cylinder sets $S(b_1, \dots, b_n) = \{\omega \in \Omega : \omega_1 = b_1, \dots, \omega_n = b_n\}$ for all sequences $(b_1, \dots, b_n) \in \{+, -\}^n$ and for all $n \geq 0$. Consider the probability space (Ω, \mathcal{F}, P) , where $P(S(b_1, \dots, b_n)) = 2^{-n}, n \geq 0$. Define a filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n, n \geq 1$ is generated by the cylinder sets $S(b_1, \dots, b_n), b_i \in \{+, -\}$.

Let $B_i, i = 1, 2, \dots$ be i.i.d. $\{+, -\}$ -valued random variables with $\Pr(B_1 = +) = \Pr(B_1 = -) = 1/2$. The random channel emerging at time n will be denoted by W^B , where $B = (B_1, B_2, \dots, B_n)$. Thus, $P(W^B = W_N^{(i)}) = 2^{-n}$ for all $i = 1, \dots, 2^n$. Let $W_n = W^B, I_n = I(W^B), Z_{\{x, x'\}, n} =$

$Z(W_{\{x,x'\}}^B)$, $Z_{v,n} = Z_v(W^B)$, and $Z_{i,n} = Z_i(W^B)$. These random variables are adapted to the above filtration (meaning that I_n etc. are measurable w.r.t. \mathcal{F}_n for every $n \geq 1$).

B. Channel polarization

In this section we state a sequence of results that shows that q -ary polar codes based on the kernel H_2 can be used to transmit reliably over the channel W for all rates $R < I(W)$.

Theorem 1: (a) Let $n \rightarrow \infty$. The random variable I_n converges a.e. to a random variable I_∞ with $E(I_\infty) = I(W)$.

(b) For all $i = 1, 2, \dots, r$

$$\lim_{n \rightarrow \infty} Z_{i,n} = Z_{i,\infty} \quad a.e.,$$

where the variables $Z_{i,\infty}$ take values 0 and 1. With probability one the vector $(Z_{i,\infty}, i = 1, \dots, r)$ takes one of the following values:

$$\begin{aligned} & (Z_{1,\infty} = 0, Z_{2,\infty} = 0, \dots, Z_{r-1,\infty} = 0, Z_{r,\infty} = 0) \\ & (Z_{1,\infty} = 1, Z_{2,\infty} = 0, \dots, Z_{r-1,\infty} = 0, Z_{r,\infty} = 0) \\ & (Z_{1,\infty} = 1, Z_{2,\infty} = 1, \dots, Z_{r-1,\infty} = 0, Z_{r,\infty} = 0) \\ & \vdots \\ & (Z_{1,\infty} = 1, Z_{2,\infty} = 1, \dots, Z_{r-1,\infty} = 1, Z_{r,\infty} = 0) \\ & (Z_{1,\infty} = 1, Z_{2,\infty} = 1, \dots, Z_{r-1,\infty} = 1, Z_{r,\infty} = 1). \end{aligned} \quad (8)$$

Let us restate part (b) of this theorem for finite n .

Proposition 1: Let $\varepsilon, \delta > 0$ be fixed. For $k = 0, 1, \dots, r$ define disjoint events

$$B_{k,n}(\varepsilon) = \left\{ \omega : (Z_{1,n}, Z_{2,n}, \dots, Z_{r,n}) \in \mathcal{R}_k \right\}$$

where $\mathcal{R}_k = \mathcal{R}_k(\varepsilon) \triangleq \left(\prod_{i=1}^k D_1 \right) \times \left(\prod_{i=k+1}^r D_0 \right)$ and $D_0 = [0, \varepsilon]$, $D_1 = (1 - \varepsilon, 1]$. Then $P(\cup_{k=0}^r B_{k,n}(\varepsilon)) \geq 1 - \delta$ starting from some $n = n(\varepsilon, \delta)$.

The proofs of these statements are given in a later part of this section.

We need the following lemma.

Lemma 1: For a DMC with q -ary input, $I(W)$ and $Z(W)$ are related by

$$I(W) \geq \log \frac{2^r}{1 + \sum_{i=1}^r 2^{i-1} Z_i(W)} \quad (9)$$

$$I(W) \leq \sum_{i=1}^r \sqrt{1 - Z_i(W)^2}. \quad (10)$$

For $r = 1$ these inequalities are proved in [1]. For $r > 1$ Eq. (9) is a restatement of [3, Prop. 3] using (7). The fact that (10) holds for all $r > 1$ is new, and is proved in the Appendix.

Inequalities (9)-(10) imply that if $(Z_1, \dots, Z_r) \in \mathcal{R}_k(\varepsilon)$ then $|I(W) - (r - k)| \leq \delta$ where $\delta \geq \max(k\sqrt{\varepsilon}, (2^{r-k} - 1)\varepsilon \log e)$.

The following proposition is an immediate corollary of the above results.

Proposition 2: (a) The random variable I_∞ is supported on the set $\{0, 1, \dots, r\}$.

(b) For every $0 \leq k \leq r$ and every $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\lim_{n \rightarrow \infty} P(\{|I_n - (r - k)| \leq \delta\} \triangle B_{k,n}(\varepsilon)) = 0.$$

(c) $E(|\{i : Z_{i,\infty} = 0\}|) = I(W)$.

Proof: The first statement is obvious from (9)-(10). To prove the second statement we note that, with the appropriate choice of ε

$$\{|I_n - (r - k)| \leq \delta\} \supset B_{k,n}(\varepsilon)$$

for all $n \geq 0$. At the same time, $P(\{|I_n - (r - k)| \leq \delta\} \cap B_{k',n}(\varepsilon)) = 0$ for all $k' \neq k$, and $P(\cup B_{k,n}(\varepsilon)) \rightarrow 1$ for any $\varepsilon > 0$. Together this implies (b). Finally, we have that $E(I_\infty) = I(W)$. Then use (a) and (b) to claim that $E(|\{i : Z_{i,\infty} = 0\}|) = \sum_{k=0}^r kP(I_\infty = k) = I(W)$. ■

We can say a bit more about the nature of convergence established in this proposition. Let us fix $k \in \{0, 1, \dots, r\}$ and define the channel for the $r - k$ rightmost bits of the transmitted symbol as follows:

$$W^{[r-k]}(y|u) = \frac{1}{2^k} \sum_{x \in \mathcal{X}: x_{k+1}^r = u} W(y|x), \quad u \in \{0, 1\}^{r-k}$$

where $x = (x_1, x_2, \dots, x_r)$.

Lemma 2: Let $V : \mathcal{X} \rightarrow \tilde{\mathcal{Y}}$ be a DMC and let $\delta > 0$. Suppose that $(Z_{1,n}(V), Z_{2,n}(V), \dots, Z_{r,n}(V)) \in \mathcal{R}_k(\varepsilon)$, for some $0 \leq k \leq r$. If ε is sufficiently small, then $I(V^{[r-k]}) \geq r - k - \delta$. In particular, it suffices to take $\varepsilon \leq 2^{-k+\delta}/(2^{r-k} - 1)$.

Proof: We may assume that $1 \leq k \leq r - 1$. Let $u \in \mathcal{X}^{r-k}$, $x = (x_1, \dots, x_k, u) \in \mathcal{X}$, $x' = (x'_1, \dots, x'_k, u) \in \mathcal{X}$. Let $v \in \{0, 1\}^{r-k} \setminus \{0\}$ and consider

$$\begin{aligned} Z(V_{\{u, u+v\}}^{[r-k]}) &= \sum_y \sqrt{V^{[r-k]}(y|u) V^{[r-k]}(y|u+v)} \\ &= \frac{1}{2^k} \sum_y \sqrt{\sum_x \sum_{x'} V(y|x) V(y|x' + v)} \\ &\leq \frac{1}{2^k} \sum_y \sum_x \sum_{x'} \sqrt{V(y|x) V(y|x' + v)} \\ &= \frac{1}{2^k} \sum_{x, x'} Z(V_{\{x, x'+v\}}) \\ &< 2^k \varepsilon \end{aligned}$$

where $v' = 0^k v_1 v_2 \dots v_{r-k}$. The last inequality follows from the fact that $Z_i(V) < \varepsilon$ for $i = k+1, \dots, r$. Since $Z_i(V^{[r-k]})$ is the average of the $Z(V_{\{u, u+v\}}^{[r-k]})$ over all v with $\text{wt}_r(v) = i$, $Z_i(V^{[r-k]}) < 2^k \varepsilon$ for all $i = 1, \dots, r - k$. Now the lemma follows from (9) in Lemma 1, ■

It turns out that the channels for individual bits converge to either perfect or fully noisy channels. If the channel for bit j is perfect then the channels for all bits $i, r \geq i > j$ are perfect. If the channel for bit i is noisy then the channels for all bits $j, 1 \leq j < i$ are noisy. The total number of near-perfect bits approaches $I(W)$. This is made formal in the next proposition.

Proposition 3: Let $\Omega_k = \{\omega : (Z_{1,\infty}, Z_{2,\infty}, \dots, Z_{r,\infty}) = 1^k 0^{r-k}\}, k = 0, 1, \dots, r$. For every $\omega \in \Omega_k$

$$\lim_{n \rightarrow \infty} |I_n - I(W_n^{[r-k]})| = 0.$$

Proof: For every $\omega \in \Omega_k$ we have that $I_n(\omega) \rightarrow r - k$. Combining this with the previous lemma and Proposition 2(b), we conclude that for such ω also $I(W_n^{[r-k]}) \rightarrow r - k$. ■

The concluding claim of this section describes the channel polarization and establishes that the total number of bits sent over almost noiseless channels approaches $NI(W)$.

Theorem 2: For any DMC $W : \mathcal{X} \rightarrow \mathcal{Y}$ the channels $W_N^{(i)}$ polarize to one of the $r + 1$ extremal configurations. Namely, let $V_i = W_N^{(i)}$ and

$$\pi_{k,N} = \frac{|\{i \in [N] : |I(V_i) - k| < \delta \wedge |I(V_i^{[k]}) - k| < \delta\}|}{N},$$

where $\delta > 0$, then $\lim_{N \rightarrow \infty} \pi_{k,N} = P(I_\infty = k)$ for all $k = 0, 1, \dots, r$. Consequently

$$\sum_{k=1}^r k \pi_k \rightarrow I(W).$$

This theorem follows directly from Theorem 1 and Propositions 2 and 3. Some examples of convergence to the extremal configurations described by this theorem are given in Sect. III below.

C. Transmission with polar codes

Let us describe a scheme of transmitting over the channel W with polar codes. Take $\varepsilon > 0$ and choose a sufficiently large n . Assume that the length of the code is $N = 2^n$. Proposition 1 implies that set $[N]$, apart from a small subset, is partitioned into $r + 1$ subsets $\mathcal{A}_{k,n}$ such that for $j \in \mathcal{A}_{k,n}$ the vector $(Z_1(W_N^{(j)}), Z_2(W_N^{(j)}), \dots, Z_r(W_N^{(j)})) \in \mathcal{R}_k(\varepsilon)$. Each $j \in \mathcal{A}_{k,n}$ refers to an r -bit symbol in which $r - k$ rightmost bits correspond to small values of $Z_i(W_N^{(j)})$. To transmit data over the channel, we write the data bits in these coordinates and encode them using the linear transformation G_N .

More specifically, let us order the coordinates $j \in [N]$ by the increase of the quantity $\sum_{i=1}^r 2^{i-1} Z_i(W_N^{(j)})$ and use these numbers to locate the subsets $\mathcal{A}_{k,n}$. We transmit data by encoding messages $u_1^N = (u_1, \dots, u_N)$ in which if $j \in \mathcal{A}_{k,n}, k = 0, \dots, r - 1$ then the symbol u_j is taken from the subset of symbols of \mathcal{X} with the first k symbols fixed and known to both the encoder and the decoder ([1] calls them frozen bits). In particular, the subset $\mathcal{A}_{r,n}$ is not used to transmit data. A polar codeword is computed as $x_1^N = u_1^N G_N$ and sent over the channel.

Decoding is performed using the “successive cancellation” procedure of [1] with the obvious constraints on the symbol values. Namely, for $j = 1, \dots, N$ put

$$\hat{u}_j = \begin{cases} u_j, & j \in \mathcal{A}_{r,n} \\ \arg \max_x W_N^{(j)}(y_1^N, \hat{u}_1^{j-1} | x), & j \in \cup_{k \leq r-1} \mathcal{A}_{k,n} \end{cases}$$

where if $j \in \mathcal{A}_{k,n}, k = 0, 1, \dots, r - 1$, then the maximum is computed over the symbols $x \in \mathcal{X}$ with the fixed (known) values of the first k bits.

The error probability of this decoding is estimated in Sect. II-E.

D. Proof of Theorem 1

Part (a) of Theorem 1 follows straightforwardly from [1], [3]. Namely, as shown in [1, Prop. 4], $I(W^+) + I(W^-) = 2I(W)$. We note that the proof in [1] uses only the fact that u_1, u_2 are recoverable from x_1, x_2 which is true in our case. Hence the sequence $I_n, n \geq 1$ forms a bounded martingale. By Doob’s theorem [13, p.196], it converges a.e. in $L^1(\Omega, \mathcal{F}, P)$ to a random variable I_∞ with $E(I_\infty) = I(W)$.

To prove part (b) we show that each of the $Z_{i,n}$ ’s converges a.s. to a $(0, 1)$ Bernoulli random variable $Z_{i,\infty}$. This convergence occurs in a concerted way in that the limit r.v.’s obey $Z_{j,\infty} \geq Z_{i,\infty}$ a.e. if $j < i$. This is shown by observing that for any fixed $i = 1, \dots, r$ and for all $v \in \mathcal{X}_i$, the $Z_{v,n}(W)$ converge to identical copies of a Bernoulli random variable.

1) Convergence of $Z_{v,n}, v \in \mathcal{X}$: In this section we shall prove that the Bhattacharyya parameters $Z_{v,n}$ converge almost surely to Bernoulli random variables. The proof forms the main technical result of this paper and is accomplished in several steps.

Lemma 3: Let

$$Z_{\max}^{(j)}(W) = \max_{v \in \mathcal{X}_j} Z_v(W), \quad j = 1, \dots, r.$$

Then

$$Z_{\max}^{(r-j)}(W^+) = Z_{\max}^{(r-j)}(W)^2, \quad j = 0, \dots, r - 1. \quad (11)$$

$$Z_{\max}^{(r)}(W^-) \leq q Z_{\max}^{(r)}(W) \quad (12)$$

$$Z_{\max}^{(r-1)}(W^-) \leq \frac{q}{2} Z_{\max}^{(r)}(W) + \frac{q}{2} Z_{\max}^{(r-1)}(W) \quad (13)$$

and generally

$$Z_{\max}^{(r-j)}(W^-) \leq \frac{q}{2} Z_{\max}^{(r)}(W) + \frac{q}{4} Z_{\max}^{(r-1)}(W) + \dots + \frac{q}{2^j} Z_{\max}^{(r-j+1)}(W) + \frac{q}{2^j} Z_{\max}^{(r-j)}(W). \quad (14)$$

Proof: In [3] it is shown that for all $v \in \mathcal{X} \setminus \{0\}$

$$Z_v(W^+) = Z_v(W)^2 \quad (15)$$

$$Z_v(W^-) \leq 2Z_v(W) + \sum_{\delta \in \mathcal{X} \setminus \{0, -v\}} Z_\delta(W) Z_{v+\delta}(W). \quad (16)$$

The first of these two equations implies (11). Now take $v \in \mathcal{X}_r$. Then in the sum on the right-hand side of (16) we have that either $\delta \in \mathcal{X}_r$ or $\delta + v \in \mathcal{X}_r$, and

$$Z_v(W^-) \leq 2Z_v(W) + (q - 2) Z_{\max}^{(r)}(W),$$

implying (12). Now take $v \in \mathcal{X}_{r-j}, j \geq 1$. The sum on δ in (16) contains $q/2$ terms with $\delta \in \mathcal{X}_r$, $q/4$ terms with $\delta \in \mathcal{X}_{r-1}$, and so on, before reaching \mathcal{X}_{r-j} . Finally, let $\delta \in \cup_{i=j}^{r-1} \mathcal{X}_{r-i} \setminus \{-v\}$. There are $(q/2^j) - 2$ possibilities, and

for each of them either $v + \delta$ or δ is in \mathcal{X}_{r-j} . This implies (14) and therefore also (13). ■

In particular, take $j = 0$. Relations (11), (12) imply that

$$Z_{\max, n+1}^{(r)} = (Z_{\max, n}^{(r)})^2 \text{ if } B_{n+1} = + \quad (17)$$

$$Z_{\max, n+1}^{(r)} \leq q Z_{\max, n}^{(r)} \text{ if } B_{n+1} = -. \quad (18)$$

Iterated random maps of this kind were studied in [14] which contains general results on their convergence and stationary distributions. We need more detailed information about this process, established in the following lemma.

Lemma 4: Let $U_n, n \geq 0$ be a sequence of random variables adapted to a filtration \mathcal{F}_n with the following properties:

- (i) $U_n \in [0, 1]$
- (ii) $P(U_{n+1} = U_n^2 | \mathcal{F}_n) \geq 1/2$
- (iii) $U_{n+1} \leq q U_n$ for some $q \in \mathbb{Z}_+$.

Then there are events Ω_0, Ω_1 such that $P(\Omega_0 \cup \Omega_1) = 1$ and $U_n(\omega) \rightarrow i$ for $\omega \in \Omega_i, i = 0, 1$.

Proof: (a) First let us rescale the process U_n so that in the neighborhood of zero it has a drift to zero. Let $\beta \in (0, 1)$ be such that

$$q^\beta - 1 < 1/4.$$

Let $X_n = U_n^\beta$. Take $\tau(\omega)$ to be the first time when $X_n(\omega) \geq 1/2$. Let $Y_n = X_{\min(n, \tau)}$. On the event $Y_n \geq 1/2$ we have $Y_n = Y_{n+1}$ or

$$E(Y_{n+1} - Y_n | \mathcal{F}_n) = 0$$

while on the event $Y_n < 1/2$ we have

$$\begin{aligned} E(Y_{n+1} - Y_n | \mathcal{F}_n) &\leq \frac{1}{2}(Y_n^2 - Y_n) + \frac{1}{2}(q^\beta Y_n - Y_n) \\ &\leq -\frac{1}{8}Y_n \leq 0. \end{aligned}$$

This implies that the sequence $Y_n, n \geq 0$ forms a supermartingale which is bounded between 0 and 1. By the convergence theorem, $Y_n \rightarrow Y_\infty$ a.e. and in $L^1(\Omega, \mathcal{F}, P)$, where Y_∞ is a random variable supported on $[0, 1]$. This implies that $EY_0 \geq EY_n \downarrow EY_\infty$. Further, if $X_0 \in [0, 1/4]$ then (since $EY_0 = EX_0$)

$$P(Y_\infty \geq 1/2) \leq 2EY_0 \leq 1/2. \quad (19)$$

(b) Now we shall prove that $P(Y_\infty \in (\delta, \frac{1}{2} - \delta)) = 0$ for any $\delta > 0$. From (ii) it follows that $P(X_{n+1} = X_n^2 | \mathcal{F}_n) \geq 1/2$, which implies that

$$P(Y_{n+1} = Y_n^2 | \mathcal{F}_n) \geq 1/2 \quad \text{on } Y_n < 1/2 \quad (20)$$

for all $n \geq 0$. Suppose that Y_∞ takes values in $(\delta, 1/2 - \delta)$ with probability $\alpha > 0$. Let $A_n = \{\omega : Y_n \in (\delta, 1/2 - \delta)\}$. Since $Y_n \rightarrow Y_\infty$ a.e., the Egorov theorem implies that there is a subset of probability arbitrarily close to $P(A_n)$ which this convergence is uniform, and thus $P(A_n) \geq \alpha/2$ for all sufficiently large n . Therefore

$$\begin{aligned} P(|Y_{n+1} - Y_n| \geq \delta^2/2) &\geq P(Y_{n+1} = Y_n^2, Y_n \in (\delta, 1/2 - \delta)) \\ &\geq \frac{\alpha}{4}, \end{aligned}$$

the last step by (20). This however contradicts the almost sure convergence of Y_n .

(c) This implies that $P(Y_\infty < 1/2) = P(Y_n \rightarrow 0) = P(U_n \rightarrow 0)$. From (19)

$$P(U_n \rightarrow 0) \geq \frac{1}{2} \quad \text{provided that } U_0 \leq \left(\frac{1}{4}\right)^{\frac{1}{\beta}}. \quad (21)$$

Moreover, if $U_0 \leq (1/2)^{1/\beta}$ then either $Y_n \rightarrow 0$ or $Y_n \geq 1/2$ for some n . This translates to

$$P((U_n \rightarrow 0) \text{ or } (U_n \geq (1/2)^{1/\beta} \text{ for some } n)) = 1 \quad (22)$$

provided that $U_0 \leq (1/2)^{1/\beta}$.

(d) Let $\delta > 0$ be such that $q(\frac{1}{2})^{\frac{1}{\beta}} < 1 - \delta$ (depending on q this may require taking a sufficiently small β). Let $L := [0, (\frac{1}{4})^{\frac{1}{\beta}}]$ and $R := [1 - \delta, 1]$. Observe that the process U_n cannot move from L to R without visiting $C := ((\frac{1}{2})^{\frac{1}{\beta}}, 1 - \delta)$. Let σ_1 be the first time when $U_n \in C$, let η_1 be the first time after σ_1 when $U_n \in L \cup R$, let σ_2 be the first time after η_1 when $U_n \in C$, etc., $\sigma_1 < \eta_1 < \sigma_2 < \eta_2 < \dots$. We shall prove that every sample path of the process eventually stays outside C , i.e., that for almost all ω there exists $k = k(\omega) < \infty$ such that $\sigma_k(\omega) = \infty$.

Assume the contrary, i.e., $\lim_{k \rightarrow \infty} P(\sigma_k < \infty) = \alpha > 0$ (since $P(\sigma_{k+1} < \infty) < P(\sigma_k < \infty)$, this limit exists.) We have

$$\begin{aligned} P(\exists k : \sigma_k = \infty) &\geq \sum_{j=1}^{\infty} P(\sigma_j \neq \infty; U_{\eta_j} \in L; \sigma_{j+1} = \infty) \\ &\geq \alpha \sum_{j=1}^{\infty} P(U_{\eta_j} \in L; \sigma_{j+1} = \infty | \sigma_j \neq \infty). \end{aligned} \quad (23)$$

Consider the process $U'_n = U_{\sigma_k + n}$ on the event $\sigma_k < \infty$ (with the measure renormalized by $P(\sigma_k < \infty)$). This process has the same properties (i)-(iii) as U_n . Let $J = \lceil \log_2(\frac{1}{\beta} \log_{1-\delta} 1/4) \rceil$, then $x^{2^J} \in L$ for any $x \in C$. Therefore, $P(U'_J \in L) \geq 2^{-J}$ by property (ii). Now consider the process U'_{J+n} on the event $U'_J \in L$. This process has properties (i)-(iii), so we can use (21) to conclude that for

$$P(U_{\eta_k} \in L; \sigma_{k+1} = \infty | \sigma_k \neq \infty) \geq 2^{-(J+1)}$$

uniformly in k . But then the sum in (23) is equal to infinity, a contradiction.

(e) The proof is completed by showing that the probability of U_n staying in $R^c = [0, 1] \setminus R$ without converging to zero is zero. We know that almost all trajectories stay outside C , so suppose that the process starts in $(0, (1/2)^{1/\beta})$. Then the probability that it enters L in a finite number of steps is uniformly bounded from below (this is shown similarly to (23)), so the probability that it does not go to L is zero. Next assume that the process starts in L , then by (22) it either goes to zero or enters C with probability one. Together with part (d) this implies that the process that starts in L converges to zero or one with probability one. ■

Lemma 5: Let $V : \mathcal{X} \rightarrow \tilde{\mathcal{Y}}$ be a channel. Let $v, v' \in \mathcal{X} \setminus \{0\}$ be such that $\text{wt}_r(v) \geq \text{wt}_r(v')$. For any $\delta' > 0$ there exists

$\delta > 0$ such that $Z_{v'}(V) \geq 1 - \delta'$ whenever $Z_v(V) \geq 1 - \delta$. In particular, we can take $\delta = \delta' q^{-3}$.

Proof: If $\text{wt}_r(v) = 1$ then $v = 10 \dots 0$, so the statement is trivial. Let $Z_v(V) \geq 1 - \delta$, where $\text{wt}_r(v) = i \geq 2$. Then for every pair $x, x' = x + v$ we have $Z(V_{\{x, x'\}}) \geq 1 - \varepsilon$, where $\varepsilon = q\delta$. Consider the unit-length vectors $z = (\sqrt{V(y|x)}, y \in \tilde{\mathcal{Y}})$, $z' = (\sqrt{V(y|x')}, y \in \tilde{\mathcal{Y}})$, and let $\theta(z, z')$ be the angle between them. We have $\cos(\theta(z, z')) = Z(V_{\{x, x'\}}) \geq 1 - \varepsilon$, and so $\|z - z'\|^2 = 2 - 2\cos(\theta(z, z')) \leq 2\varepsilon$.

Now take a pair of symbols $x_1, x_2 = x_1 + v'$ where $v' \in \mathcal{X}_s, s \leq i$. There exists a number $t \in \mathcal{X}_{r-i+s}$ such that $v' = tv$. Define $z_1 = (\sqrt{V(y|x_1)}, y \in \tilde{\mathcal{Y}})$ and $z_2 = (\sqrt{V(y|x_2)}, y \in \tilde{\mathcal{Y}})$. Let $w_j = (\sqrt{V(y|x_1 + jv)}, y \in \tilde{\mathcal{Y}}), j = 1, \dots, t-1$. From the triangle inequality

$$\begin{aligned} \|z_1 - z_2\| &\leq \|z_1 - w_1\| + \|w_1 - w_2\| + \dots + \|w_{t-1} - z_2\| \\ &\leq t\sqrt{2\varepsilon} \\ &\leq q\sqrt{2\varepsilon}. \end{aligned}$$

We obtain

$$\begin{aligned} Z(V_{\{x_1, x_2\}}) &= \cos(\theta(z_1, z_2)) = 1 - 1/2\|z_1 - z_2\|^2 \\ &\geq 1 - q^2\varepsilon \\ &= 1 - q^3\delta. \end{aligned}$$

Thus we obtain

$$Z_{v'}(V) = \frac{1}{q} \sum_x Z(V_{\{x, x+v\}}) \geq 1 - q^3\delta.$$

Remark : We can prove the previous lemma in a different way by relating the Bhattacharyya distance to the ℓ_1 -distance between $V(y|x_1)$ and $V(y|x_2)$ [15]. Then the estimate $\delta = \delta' q^{-3}$ can be improved to $\delta = \delta'(2q)^{-2}$.

Lemma 6: For all $j = 1, \dots, r$

$$Z_{\max, n}^{(j)} \xrightarrow{\text{a.e.}} Z_{\max, \infty}^{(j)}.$$

where $Z_{\max, \infty}^{(j)}$ is a Bernoulli random variable supported on $\{0, 1\}$.

Proof: For a given channel V denote

$$Z_{\max}^{[s, r]}(V) = \max(Z_{\max}^{(s)}(V), Z_{\max}^{(s+1)}(V), \dots, Z_{\max}^{(r)}(V)).$$

Eq. (15) gives us that

$$Z_{\max}^{[r-j, r]}(W^+) = (Z_{\max}^{[r-j, r]}(W))^2$$

and (14) implies that

$$Z_{\max}^{[r-j, r]}(W^-) \leq q Z_{\max}^{[r-j, r]}(W).$$

Hence by Lemma 4 the random variables $Z_{\max, \infty}^{[r-j, r]}$ are well-defined and are Bernoulli 0-1 valued a.e. for all $j = 0, 1, \dots, r-1$.

We need to prove the same for $Z_{\max, \infty}^{(r-j)}$. The proof is by induction on j . We just established the needed claim for $Z_{\max, n}^{(r)}$. For ease of understanding let us show that this implies the convergence of $Z_{\max, n}^{(r-1)}$. Indeed, $Z_{\max, \infty}^{[r-1, r]}$ is a Bernoulli 0-1

valued random variable. But so is $Z_{\max, \infty}^{(r)}$, so the possibilities are

$$(Z_{\max, \infty}^{[r-1, r]}, Z_{\max, \infty}^{(r)}) = (1, 1) \text{ or } (1, 0) \text{ or } (0, 0)$$

with probability one (note that $(0, 1)$ is ruled out by the definition of $Z_{\max}^{[r-1, r]}$). If $Z_{\max, \infty}^{(r)} = 1$ then $Z_{\max, \infty}^{(r-1)} = 1$ by Lemma 5 (this statement holds trajectory-wise). If on the other hand, the case that is realized is $(1, 0)$ then $Z_{\max, \infty}^{(r-1)} = 1$ by the definition of $Z_{\max}^{[r-1, r]}$. Finally in the case $(0, 0)$ we clearly have that $Z_{\max, \infty}^{(r-1)} = 0$, both holding trajectory-wise.

The general induction step is almost exactly the same. Assume that we have proved the required convergence for $Z_{\max}^{(r-i)}, i = 0, 1, \dots, j-1$. Assume that $Z_{\max, \infty}^{[r-j, r]} = 0$, then $Z_{\max}^{(r-j)} = 0$. If on the other hand, $Z_{\max, \infty}^{[r-j, r]} = 1$ then either one of $Z_{\max, \infty}^{(r-i)}, i < j$ equals one, and then $Z_{\max, \infty}^{(r-j)} = 1$ by Lemma 5, or $Z_{\max, \infty}^{(r-i)} = 0$ for all $i < j$, and then $Z_{\max, \infty}^{(r-j)} = 1$ by definition of $Z_{\max}^{[r-j, r]}$. ■

Now we are in a position to complete the proof of convergence.

Lemma 7: $Z_{v, n} \rightarrow Z_{v, \infty}$ a.e., where $Z_{v, \infty}$ is a $(0, 1)$ -valued random variable whose distribution depends only on the ordered weight $\text{wt}_r(v)$.

Proof: Let $\Omega_i^{(j)} = \{\omega : Z_{\max, n}^{(j)} \rightarrow i\}$, where $i = 0, 1$ and $j = 1, \dots, r$, where some of the events may be empty. For every $\omega \in \Omega_1^{(j)}, j = 1, \dots, r$ we have that for any $\delta > 0$ starting with some n_0 the quantity $Z_{\max, n}^{(j)} \geq 1 - \delta$. Thus, for $n \geq n_0$ there exists $v \in \mathcal{X}_j$, possibly depending on n , such that $Z_{v, n}(\omega) \geq 1 - \delta$. Then Lemma 5 implies that $Z_{v', n}(\omega) \geq 1 - q^3\delta$ for all $v' \in \mathcal{X}_j$, so $Z_{v, n}(\omega) \rightarrow 1$. At the same time, if $\omega \in \Omega_0^{(j)}$ then $Z_{v, n}(\omega) \rightarrow 0$ for all $v \in \mathcal{X}_j$. ■

2) Proof of Part (b) of Theorem 1:

Lemma 8: For any $i = 1, \dots, r$, the random variable $Z_{i, n}$ converges a.e. to a $(0, 1)$ -valued random variable $Z_{i, \infty}$. Moreover, $Z_{i, \infty} \geq Z_{i-1, \infty}$ a.e.

Proof: The first part follows because all the $Z_v, v \in \mathcal{X}_i$ converge to identical copies of the same random variable. Formally, Lemma 7 asserts that $Z_{v, n} \rightarrow j$ for every $v \in \mathcal{X}_i$ and every $\omega \in \Omega_j^{(i)}, j = 0, 1$. Hence taking the limit $n \rightarrow \infty$ in (6) we see that $Z_{i, n} \rightarrow j$ on $\Omega_j^{(i)}$ where $P(\Omega_0^{(i)} \cup \Omega_1^{(i)}) = 1$.

Let us prove the second part. Suppose that $Z_{i, n} \geq 1 - \varepsilon'$, then using (6) we see that $Z_{v', n} \geq 1 - 2^{i-1}\varepsilon'$ for all $v' \in \mathcal{X}_i$. Lemma 5 implies that $Z_{v, n} \geq 1 - 2^{3r+i-1}\varepsilon'$ for any $v \in \mathcal{X}$, $\text{wt}_r(v) = i$, and therefore $Z_{i, n} \geq 1 - 2^{3r+i-1}\varepsilon'$. Thus $Z_{i, n}(\omega) \rightarrow 1$ implies $Z_{i-1}(\omega) \rightarrow 1$ for all $\omega \in \Omega_1^{(i)}$ and all i . The second claim of the lemma now follows because $Z_{i, \infty}$ are 0-1 valued for all i . ■

We obtain that $Z_{i, \infty}$ is a $(0, 1)$ random variable a.e. and for all i , and if $Z_{i, \infty} = 1$ then $Z_{j, \infty} = 1$ for all $1 \leq j < i$. Consider the events $\Psi_i^{(j)} = \{\omega : Z_{j, \infty} = i\}, i = 0, 1; j =$

$1, \dots, r$. We have

$$\begin{aligned} \Psi_1^{(1)} \supset \Psi_1^{(2)} \supset \dots \supset \Psi_1^{(r)} \\ \Psi_0^{(1)} \subset \Psi_0^{(2)} \subset \dots \subset \Psi_0^{(r)}. \end{aligned}$$

We need to prove that with probability one, the vector $(Z_{i,\infty}, i = 1, \dots, r)$ takes one of the values (8). With probability one $Z_{r,\infty} = 1$ or 0. If it is equal to 1 then necessarily $Z_{r-1,\infty} = \dots = Z_{1,\infty} = 1$. Otherwise $Z_{r,\infty} = 0$. In this case it is possible that $Z_{r-1,\infty} = 1$ (in which case $Z_{r-2,\infty} = \dots = Z_{1,\infty} = 1$) or $Z_{r-1,\infty} = 0$. Of course $P(\Psi_0^{(r-1)} \cup \Psi_1^{(r-1)}) = 1$, so in particular

$$P(\Psi_0^{(r)} \setminus (\Psi_0^{(r-1)} \cup (\Psi_1^{(r-1)} \setminus \Psi_1^{(r)}))) = 0.$$

If $Z_{r-1,\infty} = 0$ then the possibilities are $Z_{r-2,\infty} = 1$ or 0, up to another event of probability 0, and so on. Thus, the union of the disjoint events given by (8) holds with probability one. Theorem 1 is proved. ■

3) *Proof of Prop. 1:* The proof is analogous to the argument in the previous paragraph. The random variable $Z_{r,n} \rightarrow Z_{r,\infty}$ a.e. . By the Egorov theorem, for any $\gamma > 0$ there are disjoint subsets $\tilde{\Psi}_0^{(r)} \subset \Psi_0^{(r)}, \tilde{\Psi}_1^{(r)} \in \Psi_1^{(r)}$ with $P(\tilde{\Psi}_0^{(r)} \cup \tilde{\Psi}_1^{(r)}) \geq 1 - \gamma$ on which this convergence is uniform. Take $n_1^{(r)}$ such that $Z_{r,n} > 1 - \varepsilon/2^{4r-1}$ for every $\omega \in \tilde{\Psi}_1^{(r)}$ and $n \geq n_1^{(r)}$. By Lemma 5 and (6) for every such ω we have $Z_{i,n} \geq 1 - \varepsilon$ for all $i = 1, \dots, r-1$; $n \geq n_1^{(r)}$. This gives rise to the event $B_{r,n}$. Otherwise let $n_0^{(r)}$ be such that $\sup_{\omega} Z_{r,n} < \varepsilon$ for $\omega \in \tilde{\Psi}_0^{(r)}$ and $n \geq n_0^{(r)}$. Consider the events $\tilde{\Psi}_0^{(r-1)} \subset \Psi_0^{(r-1)}, \tilde{\Psi}_1^{(r-1)} \subset \Psi_1^{(r-1)}$ with $P(\tilde{\Psi}_0^{(r-1)} \cup \tilde{\Psi}_1^{(r-1)}) \geq 1 - \gamma$ on which $Z_{r-1,n} \rightarrow Z_{r-1,\infty}$ uniformly. Choose $n_1^{(r-1)}$ such that $Z_{r-1,n} > 1 - \varepsilon/2^{4r-2}$ for all $n \geq n_1^{(r-1)}$ and all $\omega \in \tilde{\Psi}_1^{(r-1)}$. For every such ω we have $Z_{i,n} \geq 1 - \varepsilon$ for all $i = 1, \dots, r-2$; $n \geq n_1^{(r-1)}$. Next,

$$P(\tilde{\Psi}_0^{(r)} \setminus (\tilde{\Psi}_0^{(r-1)} \cup (\tilde{\Psi}_1^{(r-1)} \setminus \tilde{\Psi}_1^{(r)}))) \leq 2\gamma.$$

We continue in this manner until we construct all the $r+1$ events $B_{k,n}$. For this, n should be taken sufficiently large, $n \geq \max_k \max(n_0^{(k)}, n_1^{(k)})$. By taking $\gamma = \delta/r$ we can ensure that $P(\cup_k B_{k,n} \geq 1 - \delta)$. This concludes the proof.

Remark : For binary-input channels, the transmitted bits in the limit are transmitted either perfectly or carry no information about the message. Şaşıoğlu et al. [3] observed that q -ary codes constructed using Arkan's kernel H_2 share this property for transmitted symbols only if q is prime. Otherwise [3] notes the symbols can polarize to states that carry partial information about the transmission. In particular, they give an example of a quaternary-input channel $W : \{0, 1, 2, 3\} \rightarrow \{0, 1\}$ with $W(0|0) = W(0|2) = W(1|1) = W(1|3) = 1$. This channel has capacity 1 bit. Computing the channels W^+ and W^- we find that they are equivalent to the original channel W . The conclusion reached in [3] was that there are nonbinary channels that do not polarize under the action of H_2 .

We observe that the above channel corresponds to the extremal configuration 10 in (8) (the other two configurations

arise with probability 0), and therefore has to be, and is, a stable point of the channel combining operation. It is possible to reach capacity by transmitting the least significant bit of every symbol.

Paper [3] went on to show that for every $n \geq 1$ there exists a permutation $\pi_n : \mathcal{X} \rightarrow \mathcal{X}$ such that the kernels $H_2(n) : (u, v) \rightarrow (u + v, \pi_n(v))$ lead to channels that polarize to perfect or fully noisy. While the result of [3] holds for any q , in the case of $q = 2^r$ this means that configurations $00 \dots 0$ and $11 \dots 1$ arise with probability $1 - I(W)$ and $I(W)$ respectively, while all the other configurations have probability zero.

E. Rate of polarization and error probability of decoding

The following theorem, due to Arkan and Telatar [16], is useful in quantifying the rate of convergence of the channels W_n to one of the extremal configurations (8).

Theorem 3: [16] Suppose that a random process $U_n, n \geq 0$ satisfies the conditions (i)-(iii) of Lemma 4 and that (iv), U_n converges a.e. to a $\{0, 1\}$ -valued random variable U_∞ with $P(U_\infty = 0) = p$. Then for any $\alpha \in (0, 1/2)$

$$\lim_{n \rightarrow \infty} P(U_n < 2^{-2^{\alpha n}}) = p. \quad (24)$$

If condition (iii) is replaced with (iii') $U_n \leq U_{n+1}$ and $U_0 > 0$, then for any $\alpha > 1/2$,

$$\lim_{n \rightarrow \infty} P(U_n < 2^{-2^{\alpha n}}) = 0.$$

Note that, as a consequence of Lemma 4, assumption (iv) in this theorem is superfluous in that it follows from (i)-(iii).

Processes $Z_{\max,n}^{(r)}$ and $Z_{\max,n}^{[r-j,r]}, j = 0, \dots, r-1$ satisfy conditions (i)-(iii) of Lemma 4. Hence the above theorem gives the rate of convergence of each of them to zero. We argue that the convergence rate of $Z_{\max,n}^{(r-j)}, j \geq 1$ to zero is also governed by Theorem 3. Indeed, let $\Omega_i^{[r-j,r]} = \{\omega : Z_{\max,n}^{[r-j,r]} \rightarrow i\}$, $\Omega_i^{(r-j)} = \{\omega : Z_{\max,n}^{(r-j)} \rightarrow i\}$, $i = 0, 1$. Then

$$\Omega_0^{(r-j)} \supseteq \Omega_0^{[r-j,r]} \text{ and } \Omega_1^{(r-j)} = \Omega_1^{[r-j,r]} \quad (25)$$

the last equality because by Lemma 5, $Z_{\max,n}^{[r-j,r]} \rightarrow 1$ implies $Z_{\max,n}^{(r-j)} \rightarrow 1$ on every trajectory. As a consequence of (25) we have that $P(\Omega_0^{(r-j)} \setminus \Omega_0^{[r-j,r]}) = 0$. Hence $P(Z_{\max,\infty}^{(r-j)} = 0) = P(Z_{\max,\infty}^{[r-j,r]} = 0)$. Denote this common value by p_j . The random variable $Z_{\max,n}^{[r-j,r]}$ satisfies a condition of the form (24) with $p = p_j$. We obtain that for any $\alpha \in (0, 1/2)$

$$\lim_{n \rightarrow \infty} P(Z_{\max,n}^{(r-j)} < 2^{-2^{\alpha n}}) = \lim_{n \rightarrow \infty} P(Z_{\max,n}^{[r-j,r]} < 2^{-2^{\alpha n}}) = p_j.$$

Of course if $Z_{\max,n}^{(r-j)}$ is small then so is every $Z_{v,n}$ for $v \in \mathcal{X}_{r-j}$. We conclude as follows.

Proposition 4: For any $\alpha \in (0, 1/2)$ and any $v \in \mathcal{X}_j, j = 1, 2, \dots, r$

$$\lim_{n \rightarrow \infty} P(Z_{v,n} < 2^{-2^{\alpha n}}) = p_j.$$

This result enables us to estimate the probability of decoding error under successive cancellation decoding. To do this, we extend the argument of [1] to nonbinary alphabets.

The following statement follows directly from the previously established results, notably Proposition 2.

Theorem 4: Let $0 < \alpha < 1/2$. For any DMC $W : \mathcal{X} \rightarrow \mathcal{Y}$ with $I(W) > 0$ and any $R < I(W)$ there exists a sequence of r -tuples of disjoint subsets $\mathcal{A}_{0,N}, \dots, \mathcal{A}_{r-1,N}$ of $[N]$ such that $\sum_k |\mathcal{A}_{k,N}|(r-k) \geq NR$ and $Z_v(W_N^{(i)}) < 2^{-N^\alpha}$ for all $i \in \mathcal{A}_{k,N}$, all $v \in \bigcup_{l=k+1}^r \mathcal{A}_l$, and all $k = 0, 1, \dots, r-1$.

Let

$$\begin{aligned} \mathcal{E} &\triangleq \{(u_1^N, y_1^N) \in \mathcal{X}^N \times \mathcal{Y}^N : \hat{u}_1^N \neq u_1^N\} \\ \mathcal{B}_i &\triangleq \{(u_1^N, y_1^N) \in \mathcal{X}^N \times \mathcal{Y}^N : \hat{u}_1^{i-1} = u_1^{i-1}, \hat{u}_i \neq u_i\}. \end{aligned}$$

Then the block error probability of decoding is defined as

$$P_e = P(\mathcal{E}) = P\left(\bigcup_{i \in \mathcal{A}_{0,N} \cup \dots \cup \mathcal{A}_{r-1,N}} \mathcal{B}_i\right).$$

The next theorem is the main result of this section.

Theorem 5: Let $0 < \alpha < 1/2$ and let $0 < R < I(W)$, where $W : \mathcal{X} \rightarrow \mathcal{Y}$ is a DMC. The best achievable error probability of block error under successive cancellation decoding at block length $N = 2^n$ and rate R satisfies

$$P_e = O(2^{-N^\alpha}).$$

Proof: Let

$$\begin{aligned} \mathcal{E}_{i,v} &\triangleq \{(u_1^N, y_1^N) \in \mathcal{X}^N \times \mathcal{Y}^N : \\ &W_N^{(i)}(y_1^N, u_1^{i-1} | u_i) \leq W_N^{(i)}(y_1^N, u_1^{i-1} | u_i + v)\}. \end{aligned}$$

For a fixed value of $a_1^k = (a_1, a_2, \dots, a_k) \in \{0, 1\}^k$ let us define $\mathcal{X}(a_1^k) = \{x \in \mathcal{X} : x_1^k = a_1^k\}$. Notice that the decoder finds $\hat{u}_i, i \in \mathcal{A}_{k,N}$ by taking the maximum over the symbols $x \in \mathcal{X}(a_1^k)$. Then we obtain

$$\mathcal{B}_i \subseteq \bigcup_{v \in \mathcal{X}(a_1^k)} \mathcal{E}_{i,v}.$$

Using (5), we obtain

$$\begin{aligned} P(\mathcal{B}_i) &\leq \sum_{v \in \mathcal{X}(a_1^k)} P(\mathcal{E}_{i,v}) \\ &= \sum_{v \in \mathcal{X}(a_1^k)} \sum_{u_1^N, y_1^N} \frac{1}{q^N} W_N(y_1^N | u_1^N) 1_{\mathcal{E}_{i,v}}(u_1^N, y_1^N) \\ &\leq \sum_{v \in \mathcal{X}(a_1^k)} \sum_{u_1^N, y_1^N} \frac{1}{q^N} W_N(y_1^N | u_1^N) \sqrt{\frac{W_N^{(i)}(y_1^N, u_1^{i-1} | u_i + v)}{W_N^{(i)}(y_1^N, u_1^{i-1} | u_i)}} \\ &= \sum_{v \in \mathcal{X}(a_1^k)} \sum_{u_i} \frac{1}{q} Z(W_N^{(i)}_{N, \{u_i, u_i + v\}}) \\ &= \sum_{v \in \mathcal{X}(a_1^k)} Z_v(W_N^{(i)}). \end{aligned}$$

Thus the decoding error is bounded by

$$P(\mathcal{E}) \leq \sum_{i \in \mathcal{A}_{0,N} \cup \dots \cup \mathcal{A}_{r-1,N}} \sum_{v \in \mathcal{X}(a_1^k)} Z_v(W_N^{(i)}).$$

By Theorem 4, for any $R < I(W)$ there exists a sequence of r -tuples of disjoint subsets $\mathcal{A}_{0,N}, \dots, \mathcal{A}_{r-1,N}$ with $\sum_k |\mathcal{A}_{k,N}|(r-k) \geq NR$ such that

$$\sum_{i \in \mathcal{A}_{0,N} \cup \dots \cup \mathcal{A}_{r-1,N}} \sum_{v \in \mathcal{X}(a_1^k)} Z_v(W_N^{(i)}) \leq qN2^{-N^\alpha}$$

and thus we obtain that $P(\mathcal{E}) = O(2^{-N^\alpha})$. \blacksquare

III. ORDERED CHANNELS

To compute a few examples, consider “ordered symmetric channels,” called so because they provide a natural counterpart to the combinatorial definition of the ordered distance [8]. A simple example is given by the ordered erasure channel, defined as $W_r : \mathbb{F}_q^r \rightarrow (\mathbb{F}_q \cup \{?\})^r$, where

$$W_r(y|x) = \begin{cases} \varepsilon_0, & y = x, \\ \varepsilon_i, & y = (?? \dots ? x_{i+1} \dots x_r), 1 \leq i \leq r \end{cases}$$

and $W_r(y|x) = 0$ if y does not contain any erased coordinates and $y \neq x$. Its capacity equals $r - \sum_{i=1}^r i\varepsilon_i$ and is attained by sending r independent streams of data encoded for binary erasure channels with erasure probabilities $\sum_{j=i}^r \varepsilon_j, i = 1, \dots, r$. Therefore, sending r independent polar codewords over the r bit channels, one can approach the capacity of the channel.

Despite the fact that this example is trivial, it already shows the domination pattern observed in Theorem 1. Namely, it is easy to prove directly that $Z_{j,\infty} \geq Z_{i,\infty}$ a.s. for all $i > j$, thereby establishing the result of Lemma 8. For that it suffices to observe that the erasure in higher-numbered bits implies that all the lower-numbered bits are erased with probability 1. We include two examples. In Fig. 1, $r = 2$, and $\varepsilon_0 = 0.5, \varepsilon_1 = 0.4, \varepsilon_2 = 0.1$. In Fig. 2, $r = 9$ and $\varepsilon_i = 0.1, i = 0, 1, \dots, 9$. Note that the proportion of the channels with capacity $i = 0, 1, \dots, r$ bits converges to ε_i .

Another example is given by the *ordered symmetric channel* [8] which is a DMC $W : \{0, 1\}^r \rightarrow \{0, 1\}^r$ defined by the matrix $W(y|x)$ where

$$W(y|x) = 2^{-(j-1)} \varepsilon_j \quad (26)$$

for all pairs y, x such that $d_r(x, y) = j, j = 1, \dots, r$, and where $W(x|x) = \varepsilon_0$ for all $x \in \mathcal{X}$. The ordered symmetric channel models transmission over r parallel links such that, if in a given time slot a bit is received incorrectly, the bits with indices lower than it are equiprobable. This system was proposed in [19] as an abstraction of transmission in wireless fading environment. The capacity of the channel equals

$$I(W) = r + \varepsilon_0 \log_q \varepsilon_0 + \sum_{i=1}^r \varepsilon_i \log_q \left(\frac{\varepsilon_i}{q^{i-1}(q-1)} \right).$$

By Theorem 1 q -ary polar codes, $q = 2^r$ can be used to transmit at rates close to capacity on this channel; moreover, the domination pattern that emerges, exactly matches the fading nature of the bundle of r parallel channels, achieving the capacity of the system discussed above.

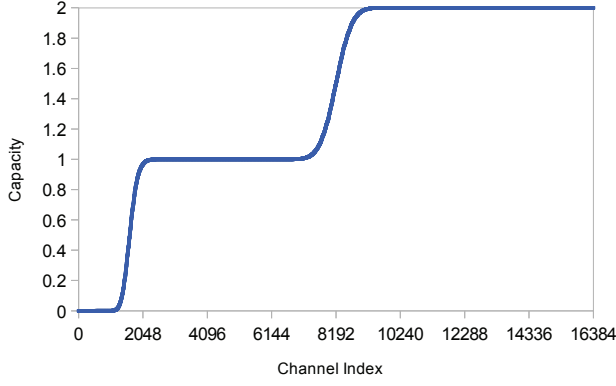


Fig. 1. 3-level polarization on the ordered erasure channel $W : \mathcal{X} \rightarrow \mathcal{Y}$, $\mathcal{X} = \{00, 01, 10, 11\}$ with transition probabilities $\varepsilon_0 := W(00|00) = 0.5$, $\varepsilon_1 := W(?x_2|x_1x_2) = 0.4$, $\varepsilon_2 := W(??|x_1, x_2) = 0.1$, for all $x_1, x_2 \in \{0, 1\}$. In this example it is easy to see that $P(I_\infty = i) = \varepsilon_i$, $i = 0, 1, 2$.

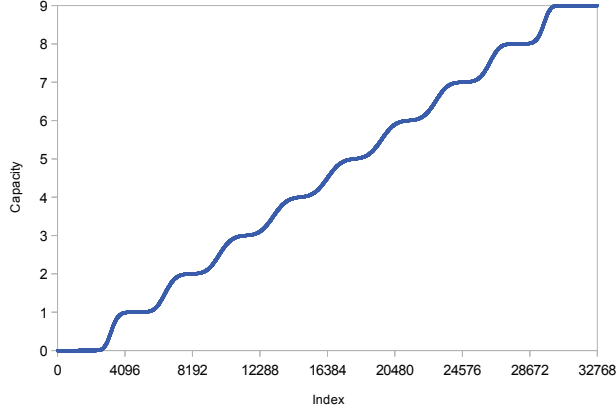


Fig. 2. 10-level polarization on the ordered erasure channel $W : \{0, 1\}^9 \rightarrow \mathcal{Y}$ with transition probabilities $\varepsilon_i = 0.1$, $i = 0, 1, \dots, 9$.

IV. CONCLUSION

The result of this paper offers more detailed information about polarization on q -ary channels, $q = 2^r$. The multilevel polarization adds flexibility to the design of the transmission scheme in that we can adjust the number of symbols that carry a given number of bits to a specified proportion of the overall transmission as long as the total number of bits is fixed. This could be useful in the design of signal constellations for coded modulation, including BICM [17], [18] as well as in other communication problems that can benefit from nonuniform symbol sets.

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APPENDIX

The proof of (10) : We shall break the expression for $I(W)$ into a sum of symmetric capacities of B-DMCs.

Let $z = (z_1, \dots, z_k)$ be an k -tuple of symbols from \mathcal{X} . Define the probability distribution $P(y|z) = \frac{1}{k} \sum_{i=1}^k W(y|z_i)$. Define a B-DMC $W_{\{z^{(1)}, z^{(2)}\}}^{(k)} : \mathcal{X}^k \rightarrow \mathcal{Y}$ with inputs $z^{(i)} \in \mathcal{X}^k$, where the transition $z^{(i)} \rightarrow y$ is given by $P(y|z^{(i)})$, $i = 1, 2$.

Lemma 9: The Bhattacharyya parameter of the channel $W_{\{z^{(1)}, z^{(2)}\}}^{(k)}$, where $z^{(1)} = (x_1, \dots, x_k)$, $z^{(2)} = (x_{k+1}, \dots, x_{2k})$, can be lower bounded by

$$Z(W_{\{z^{(1)}, z^{(2)}\}}^{(k)}) \geq \frac{1}{k} \sum_{j=1}^k Z(W_{\{x_j, x_{f(j)}\}}) \quad (27)$$

for any f which is a one-to-one mapping from the set $\{1, 2, \dots, k\}$ to $\{k+1, \dots, 2k\}$.

Proof: It suffices to prove the above inequality for some one-to-one mapping. Let $f(i) = k+i$. For brevity denote $w_{i,y} = W(y|x_i)$. We have

$$Z(W_{\{z^{(1)}, z^{(2)}\}}^{(k)}) = \frac{1}{k} \sum_y \sqrt{\left(\sum_{i=1}^k w_{i,y} \right) \left(\sum_{i=k+1}^{2k} w_{i,y} \right)},$$

while the right hand side of (27) is

$$\frac{1}{k} \sum_{j=1}^k Z(W_{\{x_j, x_{f(j)}\}}) = \frac{1}{k} \sum_y \sum_{i=1}^k \sqrt{w_{i,y} w_{k+i,y}}.$$

The Cauchy-Schwartz inequality gives us

$$\left(\sum_{i=1}^k w_{i,y} \right) \left(\sum_{i=k+1}^{2k} w_{i,y} \right) \geq \left(\sum_{i=1}^k \sqrt{w_{i,y} w_{k+i,y}} \right)^2$$

hence the lemma. \blacksquare

Let us introduce some notation. Given $z = (z_1, \dots, z_k) \in \mathcal{X}^k$, let $z \oplus x = (z_1 \oplus x, \dots, z_k \oplus x)$ where \oplus is a bit-wise modulo-2 summation. In the next lemma we consider B-DMCs $W_{\{z_m^{(1)}, z_m^{(2)}\}}^{(k)} : \mathcal{X}^k \rightarrow \mathcal{Y}$, $k = 2^{m-1}$, $m = 1, \dots, r$ with inputs of special form. Namely, $z_1^{(1)} = x_1$; $z_2^{(1)} = (x_1, x_1 \oplus x_2)$; $z_3^{(1)} = (x_1, x_1 \oplus x_2, x_1 \oplus x_3, x_1 \oplus x_2 \oplus x_3)$, and generally, $z_m^{(1)}$ is formed of x_1 plus all the possible sums of the vectors x_2, \dots, x_m with 0-1 coefficients, including the empty one. Finally, $z_m^{(2)} = z_m^{(1)} \oplus x_{m+1}$.

For $m = 0, 1, \dots, r-1$ introduce the set $\mathcal{A} = \mathcal{A}(x_1, \dots, x_{m+1}) \subset \mathcal{X}^{m+1}$ as follows:

$$\mathcal{A} = \left\{ (x_1, \dots, x_{m+1}) \in \mathcal{X}^{m+1} \mid \begin{aligned} &x_1 \in \mathcal{X}; x_2 \in \mathcal{X} \setminus \{0\}; \\ &x_j \neq \sum_{i=2}^{j-1} a_i x_i, \text{ for all choices of } a_i \in \{0, 1\}, j = 3, \dots, m+1 \end{aligned} \right\}$$

We need the following technical lemma.

Lemma 10:

$$I(W) = \sum_{m=1}^r \left(\frac{1}{2^r} \prod_{j=1}^m \frac{1}{2^r - 2^{j-1}} \right) \sum_{\mathcal{A}(x_1, \dots, x_{m+1})} I(W_{\{z_m^{(1)}, z_m^{(2)}\}}^{(k)}) \quad (28)$$

where the number k , the vectors $z_m^{(1)}, z_m^{(2)}$, and the set $\mathcal{A}(x_1, \dots, x_{m+1})$ are defined before the lemma.

Proof: First we express the capacity of W as the sum of symmetric capacities of B-DMCs.

$$\begin{aligned}
I(W) &= \frac{1}{2^r} \sum_x \sum_y W(y|x) \log \frac{W(y|x)}{P(y)} \\
&= \frac{1}{2^r} \sum_y \frac{1}{2(2^r - 1)} \sum_{x_1} \sum_{x_2: x_2 \neq 0} \left(W(y|x_1) \log \frac{W(y|x_1)}{P(y)} \right. \\
&\quad \left. + W(y|x_1 \oplus x_2) \log \frac{W(y|x_1 \oplus x_2)}{P(y)} \right) \\
&= \frac{1}{2^r(2^r - 1)} \\
&\quad \cdot \sum_y \sum_{\substack{x_1, x_2 \\ x_2 \neq 0}} \left(\frac{1}{2} W(y|x_1) \log \frac{W(y|x_1)}{\frac{1}{2}(W(y|x_1) + W(y|x_1 \oplus x_2))} \right. \\
&\quad + \frac{1}{2} W(y|x_1 \oplus x_2) \log \frac{W(y|x_1 \oplus x_2)}{\frac{1}{2}(W(y|x_1) + W(y|x_1 \oplus x_2))} \\
&\quad \left. + \frac{1}{2} (W(y|x_1) + W(y|x_1 \oplus x_2)) \right. \\
&\quad \left. \cdot \log \frac{\frac{1}{2}(W(y|x_1) + W(y|x_1 \oplus x_2))}{P(y)} \right) \\
&= \frac{1}{2^r(2^r - 1)} \left\{ \sum_{\substack{x_1, x_2 \\ x_2 \neq 0}} I(W_{\{x_1, x_1 \oplus x_2\}}) + T_2 \right\}
\end{aligned}$$

where

$$\begin{aligned}
T_2 &= \sum_y \sum_{\substack{x_1, x_2 \\ x_2 \neq 0}} \frac{1}{2} (W(y|x_1) + W(y|x_1 \oplus x_2)) \\
&\quad \cdot \log \frac{\frac{1}{2}(W(y|x_1) + W(y|x_1 \oplus x_2))}{P(y)} \}.
\end{aligned}$$

Observe that the condition $x_2 \neq 0$ is needed in order to obtain the expression for $I(W_{\{x_1, x_1 \oplus x_2\}})$.

We will apply the same technique repeatedly. In the next step we add another sum, this time on x_3 which has to satisfy the conditions $x_3 \neq 0, x_3 \neq x_2$. We have

$$\begin{aligned}
T_2 &= \sum_y \frac{1}{2(2^r - 2)} \sum_{\mathcal{A}(x_1, x_2, x_3)} \left(\frac{1}{2} (W(y|x_1) + W(y|x_1 \oplus x_2)) \right. \\
&\quad \cdot \log \frac{\frac{1}{2}(W(y|x_1) + W(y|x_1 \oplus x_2))}{P(y)} \\
&\quad + \frac{1}{2} (W(y|x_1 \oplus x_3) + W(y|x_1 \oplus x_2 \oplus x_3)) \\
&\quad \left. \cdot \log \frac{\frac{1}{2}(W(y|x_1 \oplus x_3) + W(y|x_1 \oplus x_2 \oplus x_3))}{P(y)} \right) \\
&= \frac{1}{2^r - 2} \sum_y \sum_{\mathcal{A}(x_1, x_2, x_3)} \left(\frac{1}{2} \cdot \frac{1}{2} (W(y|x_1) + W(y|x_1 \oplus x_2)) \right. \\
&\quad \left. \cdot \log \frac{\frac{1}{2}(W(y|x_1) + W(y|x_1 \oplus x_2))}{B} + B \log \frac{B}{P(y)} \right)
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{2} \cdot \frac{1}{2} (W(y|x_1 \oplus x_3) + W(y|x_1 \oplus x_2 \oplus x_3)) \\
&\cdot \log \frac{\frac{1}{2}(W(y|x_1 \oplus x_3) + W(y|x_1 \oplus x_2 \oplus x_3))}{B}
\end{aligned}$$

where $B = \frac{1}{4}(W(y|x_1) + W(y|x_1 \oplus x_2) + W(y|x_1 \oplus x_3) + W(y|x_1 \oplus x_2 \oplus x_3))$.

By now it is clear what we want to accomplish. Let us again take the sum on y inside. Recalling the definition of the channel $W^{(k)}$ before Lemma 9, we obtain

$$T_2 = \frac{1}{2^r - 2} \left\{ \sum_{\mathcal{A}(x_1, x_2, x_3)} I(W_{\{z_2^{(1)}, z_2^{(2)}\}}^{(2)}) + T_3 \right\};$$

here $I(W_{\{z_2^{(1)}, z_2^{(2)}\}}^{(2)})$ is the symmetric capacity of the B-DMC $W_{\{z_2^{(1)}, z_2^{(2)}\}}^{(2)}$ with $z_2^{(1)} = \{x_1, x_1 \oplus x_2\}$ and $z_2^{(2)} = \{x_1 \oplus x_3, x_1 \oplus x_2 \oplus x_3\}$, and T_3 is the term remaining in the expression for T_2 upon isolating this capacity:

$$T_3 = \sum_y \sum_{\mathcal{A}(x_1, x_2, x_3)} B \log \frac{B}{P(y)}.$$

Now repeat the above trick for T_3 , namely, average over all the linear combinations that this time include the vector x_4 and isolate the symmetric capacity of the channel $W^{(k)}$ that arises. Proceeding in this manner, we obtain

$$\begin{aligned}
I(W) &= \frac{1}{2^r(2^r - 1)} \sum_{\substack{x_1, x_2 \\ x_2 \neq 0}} I(W_{\{x_1, x_1 \oplus x_2\}}) \\
&+ \frac{1}{2^r(2^r - 1)(2^r - 2)} \sum_{\mathcal{A}(x_1, x_2, x_3)} I(W_{\{z_2^{(1)}, z_2^{(2)}\}}^{(2)}) \\
&+ \frac{1}{2^r(2^r - 1)(2^r - 2)} \sum_y \sum_{\mathcal{A}(x_1, x_2, x_3)} B \log \frac{B}{P(y)} \\
&= \dots \\
&= \sum_{m=1}^r \left(\frac{1}{2^r} \prod_{j=1}^m \frac{1}{2^r - 2^{j-1}} \right) \sum_{\mathcal{A}(x_1, \dots, x_{m+1})} I(W_{\{z_m^{(1)}, z_m^{(2)}\}}^{(k)})
\end{aligned}$$

where the notation $z_m^{(1)}, z_m^{(2)}, \mathcal{A}(x_1, \dots, x_{m+1})$ is introduced before the statement of lemma. ■

We continue with the proof of inequality (10). The term with $m = 1$ in (28) equals

$$\begin{aligned}
&\frac{1}{2^r(2^r - 1)} \sum_{\substack{x_1, x_2 \\ x_2 \neq 0}} I(W_{\{x_1, x_1 \oplus x_2\}}) \\
&\leq \frac{1}{2^r(2^r - 1)} \sum_{\substack{x_1, x_2 \\ x_2 \neq 0}} \sqrt{1 - Z(W_{\{x_1, x_1 \oplus x_2\}})^2} \\
&= \frac{1}{2^r(2^r - 1)} \sum_{d=1}^r \sum_{\substack{x_1, x_2 \\ \text{wt}_r(x_2)=d}} \sqrt{1 - Z(W_{\{x_1, x_1 \oplus x_2\}})^2}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2^r(2^r-1)} \sum_{d=1}^r 2^{r+d-1} \\
&\quad \cdot \sqrt{1 - \left(\frac{1}{2^{r+d-1}} \sum_{\substack{x_1, x_2 \\ \text{wt}_r(x_2)=d}} Z(W_{\{x_1, x_1 \oplus x_2\}}) \right)^2} \\
&= \frac{1}{2^r-1} \sum_{d=1}^r 2^{d-1} \sqrt{1 - Z_d^2}
\end{aligned}$$

where the first inequality is from the relation between the symmetric capacity and the Bhattacharyya parameter of B-DMCs [1], and the second inequality follows from the fact that the function $\sqrt{1-x^2}$ is concave for $0 \leq x \leq 1$.

The terms with $m \geq 2$ in (28) will be estimated using Lemma 9. We will choose the map f so that the r -vector

$$a(f) = (z^{(1)})_s \oplus (z^{(2)})_{f(s)}$$

does not depend on s . For instance, one such map is given in Lemma 9. Moreover, out of all such mappings we take the one for which $\text{wt}_r(a(f))$ is the smallest. Then the second term becomes

$$\begin{aligned}
&\frac{1}{2^r(2^r-1)(2^r-2)} \sum_{\mathcal{A}(x_1, x_2, x_3)} I(W_{\{z_2^{(1)}, z_2^{(2)}\}}) \\
&\leq \frac{1}{2^r(2^r-1)(2^r-2)} \sum_{\mathcal{A}(x_1, x_2, x_3)} \sqrt{1 - Z(W_{\{z_2^{(1)}, z_2^{(2)}\}})^2} \\
&\leq \frac{1}{2^r(2^r-1)(2^r-2)} \sum_{\mathcal{A}(x_1, x_2, x_3)} \sqrt{1 - \frac{D^2}{4}} \\
&= \frac{1}{2^r(2^r-1)(2^r-2)} \sum_{d=1}^r \sum_{\substack{\mathcal{A}(x_1, x_2, x_3) \\ \text{wt}_r(x_3)=d}} \sqrt{1 - \frac{D^2}{4}} \\
&\leq \frac{1}{2^r(2^r-1)(2^r-2)} \sum_{d=1}^r 2^r \cdot \alpha_d \\
&\quad \cdot \sqrt{1 - \left(\frac{1}{2^{r+1} \cdot \alpha_d} \sum_{\substack{\mathcal{A}(x_1, x_2, x_3) \\ \text{wt}_r(x_3)=d}} D \right)^2} \\
&\leq \frac{1}{(2^r-1)(2^r-2)} \sum_{d=1}^r \alpha_d \sqrt{1 - Z_d^2}
\end{aligned}$$

where

$$\begin{aligned}
D &= Z(W_{\{x_1, x_1 \oplus x_3\}}) + Z(W_{\{x_1 \oplus x_2, x_1 \oplus x_2 \oplus x_3\}}) \\
\alpha_d &= 2^{d-1} \cdot (2^{r+1} - 3 \cdot 2^{d-1} - 1)
\end{aligned}$$

which is the number of terms with $\text{wt}_r(x_3) = d, x_1 = 0$ under the given condition. Repeating this process, we obtain the claimed result. The full calculation is cumbersome, but its essence is captured in the example for $r = 3$ which we write out in full:

$$I(W) = \sum_{m=1}^3 \left(\frac{1}{8} \prod_{j=1}^m \frac{1}{8 - 2^{j-1}} \right) \sum_{\mathcal{A}(x_1, \dots, x_{m+1})} I(W_{\{z_m^{(1)}, z_m^{(2)}\}}^{(m)})$$

$$\begin{aligned}
&= \frac{1}{8 \cdot 7} \sum_{\mathcal{A}(x_1, x_2)} I(W_{\{x_1, x_1 \oplus x_2\}}) \\
&\quad + \frac{1}{8 \cdot 7 \cdot 6} \sum_{\mathcal{A}(x_1, x_2, x_3)} I(W_{\{z_2^{(1)}, z_2^{(2)}\}}^{(2)}) \\
&\quad + \frac{1}{8 \cdot 7 \cdot 6 \cdot 4} \sum_{\mathcal{A}(x_1, x_2, x_3, x_4)} I(W_{\{z_3^{(1)}, z_3^{(2)}\}}^{(3)}) \\
&\leq \frac{1}{7} \left(\sqrt{1 - Z_1^2} + 2\sqrt{1 - Z_2^2} + 4\sqrt{1 - Z_3^2} \right) \\
&\quad + \frac{1}{7 \cdot 6} \left(12\sqrt{1 - Z_1^2} + 18\sqrt{1 - Z_2^2} + 12\sqrt{1 - Z_3^2} \right) \\
&\quad + \frac{1}{7 \cdot 6 \cdot 4} \left(96\sqrt{1 - Z_1^2} + 48\sqrt{1 - Z_2^2} + 24\sqrt{1 - Z_3^2} \right) \\
&= \sqrt{1 - Z_1^2} + \sqrt{1 - Z_2^2} + \sqrt{1 - Z_3^2}
\end{aligned}$$

This completes the proof of (10).

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